

Signed Mahonians

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Abstract

A classical result of MacMahon gives a simple product formula for the generating function of major index over the symmetric group. A similar factorial-type product formula for the generating function of major index together with sign was given by Gessel and Simion. Several extensions are given in this paper, including a recurrence formula, a specialization at roots of unity and type B analogues.

1 Introduction

1.1 Outline

Enumeration over the symmetric group S_n and related combinatorial objects, taking into account also the *sign* of each permutation, was studied by Simion and Schmidt [36] and others (see, e.g., [35, 42, 39, 5, 28]).

The polynomial

$$\sum_{\pi \in S_n} \text{sign}(\pi) q^{\text{des}(\pi)}$$

was called the *signed Eulerian* by Désarménien and Foata [15]. An elegant formula for signed Eulerians, conjectured by Loday [24], was proved by

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Désarmenien and Foata [15] and by Wachs [41]. Type B analogues were given by Reiner [33].

MacMahon showed, about a hundred years ago, that the generating function for major index over the symmetric group has a simple product formula. The *signed Mahonian* will be defined as

$$\sum_{\pi \in S_n} \text{sign}(\pi) q^{\text{maj}(\pi)}.$$

An elegant factorial-type product formula for the signed Mahonians was given by Gessel and Simion [41, Cor. 2] (Theorem 1.3 below). Various extensions of this theorem are given in this paper.

First, a recurrence for the joint distribution of the inversion number, major index, and last digit of a permutation is given (Theorem 2.1 below). It is shown that these parameters give rise to a multiplicative, factorial-type formula, if the parameter for inversion number is set equal to 1 or to -1 (Theorem 3.2 below).

An extension in a different direction gives a factorization of the bivariate generating function of major index and inversion number at roots of unity (Theorem 4.4 below). The proof applies a remarkable identity which follows from results of Gordon [21], Roselle [34], and Foata-Schützenberger [18]. The identity was independently proved by Gessel [19, Theorem 8.5].

These extensions imply two different new proofs of Theorem 1.3.

Then Theorem 1.3 is extended to the group of signed permutations B_n , where the generating function of the flag-major index with each of the one-dimensional characters is shown to have a similar factorial type formula (Theorems 5.1, 6.1 and 6.2 below).

These results yield explicit simple generating functions for the (flag) major index on subgroups of index 2 of S_n and B_n , such as the alternating groups and the Weyl groups of type D . See Section 7.

The rest of the paper is organized as follows. Necessary background and statements of main results are given in the rest of this section. In Section 2, a multivariate recurrence formula for length, major index and last digit is proved (Theorem 2.1). Then, in Section 3, this formula is applied to prove a new extension (Theorem 3.2) of the Gessel-Simion Theorem. A second proof of the Gessel-Simion Theorem, via specialization at roots of unity, is given in Section 4. The type B analogue (Theorem 5.1) is proved in Section 5. The distribution of the (flag) major index on index 2 subgroups is then deduced in Sections 6 and 7.

1.2 Background

The Coxeter generators $\{s_i = (i, i+1) \mid 1 \leq i \leq n-1\}$ of S_n give rise to various combinatorial statistics. For $\pi \in S_n$ let the *length*, $l(\pi)$, be the standard length of π with respect to these generators, which is the same as the number of inversions of π . This notion is defined similarly for other Coxeter groups. The generating function of length in a Coxeter group W is called the *Poincaré polynomial* of W [23, Ch. 3].

For a positive integer n define

$$[n]_q := \frac{1 - q^n}{1 - q}.$$

Then

Theorem 1.1 [23, §3.15]

$$\sum_{\pi \in S_n} q^{l(\pi)} = [1]_q [2]_q \cdots [n]_q.$$

Another statistic on S_n , which has a Coxeter group interpretation, is the descent number. Given a permutation π in the symmetric group S_n , the *descent set* of π is

$$\text{Des}(\pi) := \{i \mid l(\pi) > l(\pi s_i)\} = \{i \mid \pi(i) > \pi(i+1)\}$$

and the corresponding *descent number* is $\text{des}(\pi) := |\text{Des}(\pi)|$. The *major index* of π is the following weighted enumeration of the descents

$$\text{maj}(\pi) := \sum_{i \in \text{Des}(\pi)} i.$$

A well-known classical result asserts that the length function and major index of a permutation are equidistributed over the symmetric group S_n .

Theorem 1.2 (MacMahon [25])

$$\sum_{\pi \in S_n} q^{l(\pi)} = \sum_{\pi \in S_n} q^{\text{maj}(\pi)} = [1]_q [2]_q \cdots [n]_q.$$

A similar simple factorial-type product formula for the signed Mahonians was given by Gessel and Simion [41, Cor. 2].

The *sign* of an element w in a Coxeter group W is

$$\text{sign}(w) := (-1)^{l(w)}.$$

Theorem 1.3 (The Gessel-Simion Theorem)

$$\sum_{\pi \in S_n} \text{sign}(\pi) q^{\text{maj}(\pi)} = [1]_q [2]_{-q} [3]_q [4]_{-q} \cdots [n]_{(-1)^{n-1} q}.$$

Recall that B_n denotes the group of all bijections σ of the set $[-n, n] \setminus \{0\}$ onto itself such that

$$\sigma(-a) = -\sigma(a)$$

for all $a \in [-n, n] \setminus \{0\}$, with composition as the group operation. This group is usually known as the group of “signed permutations” on $[n]$, or as the *hyperoctahedral group* of rank n , or as the classical Weyl group of type B and rank n .

It is well known (see, e.g., [8, Proposition 8.1.3]) that B_n is a Coxeter group with respect to the generating set $\{s_0, s_1, s_2, \dots, s_{n-1}\}$, where

$$s_0 := [-1, 2, \dots, n]$$

and

$$s_i := [1, 2, \dots, i-1, i+1, i, i+2, \dots, n]$$

for $i = 1, \dots, n-1$. Let $l(\sigma)$ be the standard length of $\sigma \in B_n$ with respect to its Coxeter generators.

Theorem 1.4 [23, §3.15]

$$\sum_{\pi \in B_n} q^{l(\pi)} = [2]_q [4]_q \cdots [2n]_q.$$

Despite the fact that an increasing number of enumerative results of this nature have been generalized to the hyperoctahedral group B_n (see, e.g., [9, 16, 31, 32, 37]) and that several “major index” statistics have been introduced and studied for B_n [10, 11, 12, 17, 29, 30, 40] no generalization of MacMahon’s result to B_n has been found until a new statistic, the *flag major index*, was introduced.

The *flag-major* of $\sigma \in B_n$ is defined as

$$\text{flag-major}(\sigma) := 2 \cdot \text{maj}(\sigma) + \text{neg}(\sigma)$$

where

$$\text{neg}(\sigma) := \#\{1 \leq i \leq n \mid \sigma(i) < 0\}$$

and $\text{maj}(\sigma)$ is the major index of the sequence $(\sigma(1), \dots, \sigma(n))$, with respect to the order

$$-1 < \dots < -n < 1 < \dots < n.$$

A type B analogue of Theorem 1.2 was given in [4].

Theorem 1.5 [4]

$$\sum_{\pi \in B_n} q^{l(\pi)} = \sum_{\pi \in B_n} q^{\text{flag-major}(\pi)} = [2]_q [4]_q \cdots [2n]_q.$$

For a unified definition of the classical major index and the flag-major index as a length of a distinguished canonical expression see [4]. The flag-major index has many combinatorial and algebraic properties which are shared with the classical major index on S_n [4, 1, 22, 2, 3, 7, 13]. In this paper we will give a type B analogue of the Gessel-Simion Theorem (Theorem 5.1 below), as well as other new extensions of this theorem.

1.3 Main Results

We find a recurrence (theorem 2.1 below) for the joint distribution of length, major index, and last digit, which leads to the following result. Let

$$\text{last}(\pi) := \pi(n) - 1.$$

Then

Theorem 1.6 (see Theorem 3.2 below)

For $\varepsilon = \pm 1$,

$$\sum_{\pi \in S_n} \varepsilon^{l(\pi)} q^{\text{maj}(\pi)} z^{\text{last}(\pi)} = [1]_q \cdot [2]_{\varepsilon q} \cdot [3]_q \cdot [4]_{\varepsilon q} \cdots [n-1]_{\pm q} \cdot [n]_{\pm \varepsilon q/z} \cdot z^{n-1}.$$

This theorem shows that the distribution of (signed) major index over permutations with prescribed last digit is essentially independent of this digit (Corollary 3.4). Letting $\varepsilon = -1$ and $z = 1$ gives Theorem 1.3.

A second new proof of Theorem 1.3 uses a known identity (Theorem 4.3 below) involving the generating function for length and major index. This also leads to a factorization at roots of unity other than ± 1 .

Let

$$A_n(t, q) := \sum_{\pi \in S_n} t^{l(\pi)} q^{\text{maj}(\pi)}.$$

For a positive integer n define

$$(q)_n := (1 - q)(1 - q^2) \cdots (1 - q^n).$$

Theorem 1.7 (see Theorem 4.4 below)

Let n and m be positive integers. Let ζ be a primitive m th root of unity and assume that $n = mk + i$ with $0 \leq i < m$. Then

$$A_n(\zeta, q) = A_i(\zeta, q) \frac{(q)_n}{(q)_i (1 - q^m)^k}.$$

The case $m = 2$ gives Theorem 1.3.

A type B analogue of Theorem 1.3 is :

Theorem 1.8 (see Theorem 5.1 below)

$$\sum_{\pi \in B_n} \text{sign}(\pi) \cdot q^{\text{flag-major}(\pi)} = [2]_{-q} [4]_q \cdots [2n]_{(-1)^n q}.$$

Explicit generating functions of the major index and flag major index on distinguished subgroups follow from Theorems 1.3 and 1.8. See Corollaries 7.1 and 7.2 below.

2 A Recurrence Formula

Let S_n be the symmetric group. For $\pi \in S_n$ define the following statistics:

$$\begin{aligned} \text{inv}(\pi) &:= \text{inversion number of } \pi \\ & (= \text{length of } \pi \text{ w.r.t. the usual Coxeter generators of } S_n) \\ \text{maj}(\pi) &:= \text{major index of } \pi = \sum \{ 1 \leq i \leq n-1 \mid \pi(i) > \pi(i+1) \} \\ \text{last}(\pi) &:= \pi(n) - 1, \text{ one less than the last digit in } \pi \end{aligned}$$

Define the multivariate generating function

$$f_n(x, y, z) := \sum_{\pi \in S_n} x^{\text{inv}(\pi)} y^{\text{maj}(\pi)} z^{\text{last}(\pi)}. \quad (1)$$

Theorem 2.1 (Recurrence Formula)

$$f_1(x, y, z) = 1$$

and, for $n \geq 2$,

$$\begin{aligned} (x - z)f_n(x, y, z) &= (x^n y^{n-1} - z^n) \cdot f_{n-1}(x, y, 1) \\ &\quad + x^{n-1} (1 - y^{n-1}) z \cdot f_{n-1}(x, y, z/x). \end{aligned}$$

Proof. The case $n = 1$ is clear. Assume $n \geq 2$.

Given a permutation

$$\pi = (\pi(1), \dots, \pi(n-1)) \in S_{n-1},$$

append k ($1 \leq k \leq n$) as the n th digit, while adding 1 to each existing digit between k and $n-1$, to get a permutation

$$\bar{\pi} = (\bar{\pi}(1), \dots, \bar{\pi}(n-1), k) \in S_n$$

where, for $1 \leq i \leq n-1$,

$$\bar{\pi}(i) = \begin{cases} \pi(i), & \text{if } \pi(i) < k; \\ \pi(i) + 1, & \text{otherwise.} \end{cases}$$

The new statistics for $\bar{\pi}$ are:

$$\begin{aligned} \text{inv}(\bar{\pi}) &= \text{inv}(\pi) + (n - k) \\ \text{maj}(\bar{\pi}) &= \begin{cases} \text{maj}(\pi), & \text{if } k > \pi(n-1); \\ \text{maj}(\pi) + (n-1), & \text{otherwise.} \end{cases} \\ \text{last}(\bar{\pi}) &= k - 1 \end{aligned}$$

We can therefore compute

$$\begin{aligned} f_n &= f_n(x, y, z) \\ &= \sum_{\pi \in S_{n-1}} \sum_{k=1}^n x^{\text{inv}(\bar{\pi})} y^{\text{maj}(\bar{\pi})} z^{\text{last}(\bar{\pi})} \\ &= \sum_{\pi \in S_{n-1}} x^{\text{inv}(\pi) + n - 1} y^{\text{maj}(\pi)} \\ &\quad \times \left[y^{n-1} \sum_{k=1}^{\text{last}(\pi)+1} x^{1-k} z^{k-1} + \sum_{k=\text{last}(\pi)+2}^n x^{1-k} z^{k-1} \right] \\ &= (1 - z/x)^{-1} \sum_{\pi \in S_{n-1}} x^{\text{inv}(\pi) + n - 1} y^{\text{maj}(\pi)} \\ &\quad \times \left[y^{n-1} \left(1 - (z/x)^{\text{last}(\pi)+1} \right) + \left((z/x)^{\text{last}(\pi)+1} - (z/x)^n \right) \right] \\ &= (1 - z/x)^{-1} [(x^{n-1} y^{n-1} - x^{-1} z^n) f_{n-1}(x, y, 1) \\ &\quad + x^{n-2} (1 - y^{n-1}) z f_{n-1}(x, y, z/x)]. \end{aligned}$$

Multiplying both sides by $x - z$ gives the claimed recurrence.

□

3 A Multiplicative Generating Function

In general, the generating function from the previous section is a complicated polynomial of its variables. However, assuming in addition that $x^2 = 1$ leads to surprisingly simple results.

Corollary 3.1 *The first few values of f_n , assuming $x = \varepsilon = \pm 1$, are:*

$$\begin{aligned} f_1(\varepsilon, q, z) &= 1 \\ f_2(\varepsilon, q, z) &= z + \varepsilon q \\ f_3(\varepsilon, q, z) &= (1 + \varepsilon q)(z^2 + qz + q^2) \\ f_4(\varepsilon, q, z) &= (1 + \varepsilon q)(1 + q + q^2)(z^3 + \varepsilon qz^2 + q^2z + \varepsilon q^3) \end{aligned}$$

The case $\varepsilon = z = 1$ is a well-known result of MacMahon.

Theorem 3.2 *For $\varepsilon = \pm 1$,*

$$\begin{aligned} \sum_{\pi \in S_n} \varepsilon^{\text{inv}(\pi)} q^{\text{maj}(\pi)} z^{\text{last}(\pi)} &= \left(\prod_{i=1}^{n-1} [i]_{\varepsilon^{i-1}q} \right) \cdot [n]_{\varepsilon^{n-1}q/z} \cdot z^{n-1} \\ &= [1]_q [2]_{\varepsilon q} [3]_q [4]_{\varepsilon q} \cdots [n-1]_{\pm q} \cdot [n]_{\pm \varepsilon q/z} z^{n-1}. \end{aligned}$$

Proof. By induction on n . By Corollary 3.1, the claim is true for $n = 1$ (as well as for $n = 2, 3, 4$). Assume now that the claim holds for $n - 1$, where $n \geq 2$. Thus

$$f_{n-1}(\varepsilon, q, z) = \left(\prod_{i=1}^{n-2} [i]_{\varepsilon^{i-1}q} \right) \cdot [n-1]_{\varepsilon^{n-2}q/z} \cdot z^{n-2}.$$

Substituting in the recurrence formula of Theorem 2.1 and eliminating the factor

$$\left(\prod_{i=1}^{n-2} [i]_{\varepsilon^{i-1}q} \right),$$

it remains to show that

$$\begin{aligned} &(\varepsilon - z)[n-1]_{\varepsilon^{n-2}q} [n]_{\varepsilon^{n-1}q/z} \cdot z^{n-1} \\ &= (\varepsilon^n q^{n-1} - z^n)[n-1]_{\varepsilon^{n-2}q} + \varepsilon^{n-1}(1 - q^{n-1})z[n-1]_{\varepsilon^{n-1}q/z} \cdot (z/\varepsilon)^{n-2}. \end{aligned}$$

Using the definition of $[k]_q$, this is equivalent to

$$\begin{aligned} &\frac{(\varepsilon - z)(1 - (\varepsilon^{n-2}q)^{n-1})(z^n - (\varepsilon^{n-1}q)^n)}{(1 - \varepsilon^{n-2}q)(z - \varepsilon^{n-1}q)} \\ &= \frac{(\varepsilon^n q^{n-1} - z^n)(1 - (\varepsilon^{n-2}q)^{n-1})}{1 - \varepsilon^{n-2}q} + \frac{\varepsilon z(1 - q^{n-1})(z^{n-1} - (\varepsilon^{n-1}q)^{n-1})}{z - \varepsilon^{n-1}q}. \end{aligned}$$

Clearing denominators and using the fact that $(n-2)(n-1)$ is even, we can transform this equation into

$$\begin{aligned} (\varepsilon - z)(1 - q^{n-1})(z^n - q^n) &= (\varepsilon^n q^{n-1} - z^n)(1 - q^{n-1})(z - \varepsilon^{n-1}q) \\ &+ \varepsilon z(1 - q^{n-1})(z^{n-1} - \varepsilon^{n-1}q^{n-1})(1 - \varepsilon^{n-2}q). \end{aligned}$$

Dividing by $(1 - q^{n-1})$ one gets

$$(\varepsilon - z)(z^n - q^n) = (\varepsilon^n q^{n-1} - z^n)(z - \varepsilon^{n-1}q) + \varepsilon z(z^{n-1} - \varepsilon^{n-1}q^{n-1})(1 - \varepsilon^n q),$$

completing the proof. □

Letting $z = 1$, one gets

Corollary 3.3

$$\begin{aligned} \sum_{\pi \in S_n} q^{\text{maj}(\pi)} &= [n]_q! := [1]_q [2]_q \cdots [n]_q \\ \sum_{\pi \in S_n} \text{sign}(\pi) q^{\text{maj}(\pi)} &= [n]_{\pm q}! := [1]_q [2]_{-q} [3]_q [4]_{-q} \cdots [n]_{(-1)^{n-1}q} \end{aligned}$$

The first formula is a classical result of MacMahon [25], and the second was first proved by Gessel and Simion [41, Cor. 2].

Corollary 3.4 *The distributions of maj and of maj with sign over all permutations with a prescribed last digit are essentially independent of this digit, namely: if*

$$S_n(k) := \{ \pi \in S_n \mid \pi(n) = k \} \quad (1 \leq k \leq n)$$

then, for $\varepsilon = \pm 1$,

$$\begin{aligned} \sum_{\pi \in S_n(k)} \varepsilon^{\text{inv}(\pi)} q^{\text{maj}(\pi)} &= f_{n-1}(\varepsilon, q, 1) \cdot (\varepsilon^{n-1}q)^{n-k} \\ &= \left(\prod_{i=1}^{n-1} [i]_{\varepsilon^{i-1}q} \right) \cdot (\varepsilon^{n-1}q)^{n-k}. \end{aligned}$$

4 Specialization at Roots of Unity

A proof of Theorem 1.7 is given in this section.

Suppose that we have a sequence $f_0(q), f_1(q), \dots$ of polynomials in q defined by the Eulerian generating function

$$F(u; q) = \sum_{n=0}^{\infty} f_n(q) \frac{u^n}{(q)_n}, \quad (2)$$

where $(q)_n := (1-q)(1-q^2) \cdots (1-q^n)$. We would like to study the values of $f_n(q)$ at a root of unity. We cannot simply evaluate (2) at a root of unity, since this would make denominators vanish. Instead we take a less direct approach.

Fix a positive integer m , and let $\phi_m(q)$ be the cyclotomic polynomial of order m in q (whose roots are all the primitive m th roots of unity). If $f(q)$ and $g(q)$ are polynomials in q with rational coefficients and ζ is a primitive m th root of unity, then $f(q) \equiv g(q) \pmod{\phi_m(q)}$ if and only if $f(\zeta) = g(\zeta)$, since $\phi_m(q)$ is irreducible over the rationals and $\phi_m(\zeta) = 0$.

Given two Eulerian generating functions $F(u; q) = \sum_{n=0}^{\infty} f_n(q) u^n / (q)_n$ and $G(u; q) = \sum_{n=0}^{\infty} g_n(q) u^n / (q)_n$, by $F(u; q) \equiv G(u; q)$ we mean that $f_n(q) \equiv g_n(q) \pmod{\phi_m(q)}$ for all n . Henceforth we take all congruences to be modulo $\phi_m(q)$.

The basic facts about these congruences are contained in the following lemma:

Lemma 4.1 *Let $u_i := u^i / (q)_i$.*

(i) *If $0 \leq i, j < m$ and $i + j \geq m$ then $u_i u_j \equiv 0$.*

(ii) *If $0 \leq i < m$ then*

$$u_{mk+i} \equiv \frac{u_m^k}{k!} u_i.$$

Proof. Let ζ be a primitive m th root of unity. For (i), we have

$$u_i u_j = \frac{u^{i+j}}{(q)_i (q)_j} = \frac{(q)_{i+j}}{(q)_i (q)_j} u_{i+j}.$$

The quotient in the right-hand-side is a polynomial in q (a q -binomial coefficient, see below). Since $(q)_{i+j}$ vanishes for $q = \zeta$ but $(q)_i (q)_j$ does not, (i) follows.

For (ii), we have

$$\frac{u_m^k}{k!} u_i = \frac{u^{mk+i}}{(q)_m^k k! (q)_i} = \frac{(q)_{mk+i}}{(q)_m^k k! (q)_i} u_{mk+i},$$

so it suffices to show that

$$\left. \frac{(q)_{mk+i}}{(q)_m^k k! (q)_i} \right|_{q=\zeta} = 1.$$

To prove this, we show that

$$\left. \frac{(q)_{mk+i}}{(q)_{mk} (q)_i} \right|_{q=\zeta} = 1$$

and that

$$\left. \frac{(q)_{mk}}{(q)_m^k} \right|_{q=\zeta} = k!.$$

For the first equality, we have

$$\frac{(q)_{mk+i}}{(q)_{mk} (q)_i} = \frac{1 - q^{mk+1}}{1 - q} \frac{1 - q^{mk+2}}{1 - q^2} \cdots \frac{1 - q^{mk+i}}{1 - q^i}.$$

Since $\zeta^{mk+j} = \zeta^j \neq 1$ for $j = 1, 2, \dots, i$, the equality follows.

For the second equality, let us write

$$(q)_{mk} = \prod_{\substack{1 \leq l \leq mk \\ m \nmid l}} (1 - q^l) \cdot \prod_{j=1}^k (1 - q^{mj}),$$

so

$$\frac{(q)_{mk}}{(q)_m^k} = \frac{\prod_{1 \leq l \leq mk, m \nmid l} (1 - q^l)}{(q)_{m-1}^k} \cdot \prod_{j=1}^k \frac{1 - q^{mj}}{1 - q^m}.$$

We may evaluate the first factor on the right at $q = \zeta$ by simply setting $q = \zeta$, since neither the numerator nor the denominator vanishes, and we see easily that this factor becomes 1. Writing the second factor as

$$\prod_{j=1}^k (1 + q^m + \cdots + q^{m(j-1)}),$$

we see that setting $q = \zeta$ in it yields $k!$. □

Now recall that the q -binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the polynomial in q defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q)_n}{(q)_k (q)_{n-k}}$$

for $0 \leq k \leq n$, with $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ for $n < k$. As a consequence of Lemma 4.1 we obtain a frequently rediscovered result of Gloria Olive [26, (1.2.4)] about the evaluation of q -binomial coefficients at roots of unity:

Corollary 4.2 *Let m be a positive integer and let ζ be a primitive m th root of unity. Let a_1, a_2, b_1 , and b_2 be nonnegative integers with $0 \leq b_1, b_2 < m$. Then*

$$\begin{bmatrix} (ma_1 + b_1) + (ma_2 + b_2) \\ ma_1 + b_1 \end{bmatrix}_\zeta = \begin{pmatrix} a_1 + a_2 \\ a_1 \end{pmatrix} \begin{bmatrix} b_1 + b_2 \\ b_1 \end{bmatrix}_\zeta.$$

Proof. With the notation of Lemma 4.1 we have

$$\begin{aligned} \begin{bmatrix} (ma_1 + b_1) + (ma_2 + b_2) \\ ma_1 + b_1 \end{bmatrix}_q u_{(ma_1 + b_1) + (ma_2 + b_2)} &= \frac{u^{ma_1 + b_1}}{(q)_{ma_1 + b_1}} \frac{u^{ma_2 + b_2}}{(q)_{ma_2 + b_2}} \\ &= u_{ma_1 + b_1} u_{ma_2 + b_2}. \end{aligned}$$

By Lemma 4.1(ii) this is congruent modulo $\phi_m(q)$ to

$$\frac{u_m^{a_1}}{a_1!} u_{b_1} \frac{u_m^{a_2}}{a_2!} u_{b_2}.$$

If $b_1 + b_2 \geq m$ then, by Lemma 4.1(i), this is congruent to 0. Otherwise we have, by Lemma 4.1(ii),

$$\begin{aligned} \frac{u_m^{a_1}}{a_1!} u_{b_1} \frac{u_m^{a_2}}{a_2!} u_{b_2} &= \begin{pmatrix} a_1 + a_2 \\ a_1 \end{pmatrix} \frac{u_m^{a_1 + a_2}}{(a_1 + a_2)!} \cdot \begin{bmatrix} b_1 + b_2 \\ b_1 \end{bmatrix}_q u_{b_1 + b_2} \\ &\equiv \begin{pmatrix} a_1 + a_2 \\ a_1 \end{pmatrix} \begin{bmatrix} b_1 + b_2 \\ b_1 \end{bmatrix}_q u_{m(a_1 + a_2) + (b_1 + b_2)}, \end{aligned}$$

and the result follows. \square

Our proof of Theorem 1.7 is based on the generating function for the bivariate distribution of length and major index:

Theorem 4.3 *Let the polynomials $A_n(q, r)$ be defined by*

$$A(u; q) := \prod_{i,j=0}^{\infty} \frac{1}{1 - q^i r^j u} = \sum_{n=0}^{\infty} \frac{A_n(q, r)}{(q)_n (r)_n} u^n. \quad (3)$$

Then

$$A_n(q, r) = \sum_{\pi \in S_n} q^{l(\pi)} r^{\text{maj}(\pi)}.$$

Historical Note: Theorem 4.3 was first proved by Gessel [19, Theorem 8.5]. (For a refinement that also includes the number of descents, see [20].) Basil Gordon [21] had earlier given a combinatorial interpretation to the coefficients of $A_n(q, r)$, but did not describe it very explicitly. (In fact, he considered the generalization $\prod_{i,j,\dots,k=0}^{\infty} (1 - q^i r^j \dots s^k u)^{-1}$.) D. P. Roselle [34] explained Gordon's combinatorial interpretation more explicitly. His result is equivalent to

$$A_n(q, r) = \sum_{\pi \in S_n} q^{\text{maj}(\pi^{-1})} r^{\text{maj}(\pi)}.$$

Then D. Foata and M.-P. Schützenberger [18] gave a bijective proof that

$$\sum_{\pi \in S_n} q^{\text{maj}(\pi^{-1})} r^{\text{maj}(\pi)} = \sum_{\pi \in S_n} q^{l(\pi)} r^{\text{maj}(\pi)},$$

which, together with the result of Gordon and Roselle, implies Theorem 4.3.

Theorem 4.4 *Let n and m be positive integers. Let ζ be a primitive m th root of unity, and assume that $n = mk + i$ with $0 \leq i < m$. Then*

$$A_n(\zeta, r) = A_i(\zeta, r) \frac{(r)_n}{(r)_i (1 - r^m)^k}.$$

Proof. To find a congruence modulo $\phi_m(q)$ for the polynomials $A_n(q, r)$, think of (3) as an Eulerian generating function in which the coefficient of $u^n / (q)_n$ is $A_n(q, r) / (r)_n$. By taking logarithms and exponentiating, we see that

$$\begin{aligned} A(u; q) &= \prod_{i,j=0}^{\infty} \frac{1}{1 - q^i r^j u} = \exp \left(- \sum_{i,j=0}^{\infty} \ln(1 - q^i r^j u) \right) \\ &= \exp \left(\sum_{i,j=0}^{\infty} \sum_{t=1}^{\infty} \frac{(q^i r^j u)^t}{t} \right) = \exp \left(\sum_{t=1}^{\infty} \frac{u^t}{t(1 - q^t)(1 - r^t)} \right). \end{aligned}$$

Now

$$\sum_{t=1}^{\infty} \frac{u^t}{t(1-q^t)(1-r^t)} = \sum_{t=1}^{\infty} \frac{(q)_{t-1}}{t(1-r^t)} \frac{u^t}{(q)_t} \equiv \sum_{t=1}^m \frac{(q)_{t-1}}{t(1-r^t)} \frac{u^t}{(q)_t},$$

so

$$A(u; q) \equiv \exp\left(\sum_{t=1}^{m-1} \frac{(q)_{t-1}}{t(1-r^t)} u_t\right) \cdot \exp\left(\frac{(q)_{m-1}}{m(1-r^m)} u_m\right).$$

Using Lemma 4.1(i) we see that

$$\exp\left(\sum_{t=1}^{m-1} \frac{(q)_{t-1}}{t(1-r^t)} u_t\right) \equiv \sum_{i=0}^{m-1} B_i(q, r) u_i,$$

where $B_i(q, r)$ are polynomials in q whose coefficients are rational functions of r .

Now let ζ be a primitive m th root of unity. Setting $x = 1$ in

$$(1 - \zeta x) \cdots (1 - \zeta^{m-1} x) = (1 - x^m)/(1 - x) = 1 + x + \cdots + x^{m-1}$$

we see that

$$(1 - \zeta) \cdots (1 - \zeta^{m-1}) = m.$$

Thus $(q)_{m-1} \equiv m$, so with the terminology of Lemma 4.1 we have

$$\frac{(q)_{m-1}}{m(1-r^m)} u_m \equiv \frac{u_m}{1-r^m}$$

and

$$\exp\left(\frac{(q)_{m-1}}{m(1-r^m)} u_m\right) \equiv \exp\left(\frac{u_m}{1-r^m}\right) = \sum_{k=0}^{\infty} \frac{u_m^k}{k! (1-r^m)^k}.$$

It follows that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{A_n(q, r)}{(r)_n} \frac{u^n}{(q)_n} &\equiv \sum_{i=0}^{m-1} \sum_{k=0}^{\infty} B_i(q, r) \frac{u_i u_m^k}{k! (1-r^m)^k} \\ &\equiv \sum_{i=0}^{m-1} \sum_{k=0}^{\infty} \frac{B_i(q, r)}{(1-r^m)^k} \frac{u^{mk+i}}{(q)_{mk+i}}, \end{aligned}$$

by Lemma 4.1(ii). Thus, if $n = mk + i$ with $0 \leq i < m$, then

$$\frac{A_n(q, r)}{(r)_n} \equiv \frac{B_i(q, r)}{(1-r^m)^k}$$

or equivalently

$$\frac{A_n(\zeta, r)}{(r)_n} = \frac{B_i(\zeta, r)}{(1 - r^m)^k}.$$

Letting $k = 0$ (so that $n = i$) we get

$$B_i(\zeta, r) = \frac{A_i(\zeta, r)}{(r)_i} \quad (0 \leq i < m)$$

and the result follows. \square

Second Proof of Theorem 1.3. Take $m = 2$ in Theorem 4.4 and simplify. \square

For some other results involving the evaluation of $A_n(q, r)$ at roots of unity, see [6] and [21].

5 A Signed Mahonian for B_n

Let B_n be the hyperoctahedral group. The *flag-major* of $\sigma \in B_n$ is defined as

$$\text{flag-major}(\sigma) := 2 \text{maj}(\sigma) + \text{neg}(\sigma),$$

where

$$\text{neg}(\sigma) := \#\{i \mid \sigma(i) < 0\}$$

and $\text{maj}(\sigma)$ is the major index of the sequence $(\sigma(1), \dots, \sigma(n))$, with respect to the order

$$-1 < \dots < -n < 1 < \dots < n.$$

Recall that for every $\sigma \in B_n$ we define

$$\text{sign}(\sigma) = (-1)^{l(\sigma)},$$

where the length l (here and throughout this section) is taken with respect to the Coxeter generators of B_n .

Theorem 5.1

$$\sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{flag-major}(\sigma)} = [2]_{-q} [4]_q \cdots [2n]_{(-1)^n q}.$$

Remark 5.2 *The above order appeared in [4]. In [1] we considered another natural order :*

$$-n < \cdots < -1 < 1 < \cdots < n.$$

The distribution of flag-major is the the same for both orders, but the joint distribution of flag-major and length is different, and Theorem 5.1 does not hold for flag-major defined with respect to the latter order. It was shown in [4] that flag-major defined with respect to the first order satisfies some further remarkable properties (e.g., it is the length of a certain decomposition of the permutation). These properties do not hold for the second order.

Proof. We use the decomposition

$$B_n = U_n \cdot S_n,$$

where

$$U_n := \{ \tau \in B_n \mid \tau(1) < \cdots < \tau(n) \}$$

with respect to the order

$$-1 < \cdots < -n < 1 < \cdots < n,$$

and

$$S_n := \{ \pi \in B_n \mid \text{neg}(\pi) = 0 \}.$$

This decomposition appeared in [1] (where it was taken with respect to the other order).

Note that every $\sigma \in B_n$ has a unique decomposition $\sigma = \tau\pi$, $\tau \in U_n$, $\pi \in S_n$. Then, by definition,

$$\text{flag-major}(\sigma) = 2 \cdot \text{maj}(\pi) + \text{neg}(\tau).$$

Thus

$$\begin{aligned} \sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{flag-major}(\sigma)} &= \sum_{\tau \in U_n, \pi \in S_n} \text{sign}(\tau\pi) q^{2 \cdot \text{maj}(\pi) + \text{neg}(\tau)} \\ &= \sum_{\tau \in U_n} \text{sign}(\tau) q^{\text{neg}(\tau)} \cdot \sum_{\pi \in S_n} \text{sign}(\pi) q^{2 \cdot \text{maj}(\pi)}. \end{aligned}$$

By Corollary 3.3, the second factor is equal to

$$\sum_{\pi \in S_n} \text{sign}(\pi) q^{2 \cdot \text{maj}(\pi)} = [1]_{q^2} [2]_{-q^2} \cdots [n]_{\pm q^2}.$$

We shall compute the first factor. Define

$$U_n(k) := \{ \tau \in U_n \mid \text{neg}(\tau) = k \} \quad (0 \leq k \leq n).$$

Then

$$\sum_{\tau \in U_n} \text{sign}(\tau) q^{\text{neg}(\tau)} = \sum_{k=0}^n \sum_{\tau \in U_n(k)} \text{sign}(\tau) \cdot q^k = \sum_{k=0}^n q^k \sum_{\tau \in U_n(k)} (-1)^{l(\tau)}.$$

Recall from [9, Proposition 3.1 and Corollary 3.2] [8, Propositions 8.1.1 and 8.1.2] that for every $\sigma \in B_n$

$$l(\sigma) = \text{inv}(\sigma) + \sum_{\{1 \leq i \leq n \mid \sigma(i) < 0\}} |\sigma(i)|,$$

where $\text{inv}(\sigma)$ is taken with respect to the order

$$-n < \cdots < -1 < 1 < \cdots < n.$$

Now U_n consists of all elements whose entries are increasing with respect to the order $-1 < \cdots < -n < 1 < \cdots < n$. Thus for every $\tau \in U_n(k)$

$$\text{inv}(\tau) = \binom{k}{2}$$

and

$$l(\tau) = \binom{k}{2} + \sum_{i=1}^k |\tau(i)|.$$

It follows that

$$\begin{aligned} \sum_{\tau \in U_n(k)} (-1)^{l(\tau)} &= \sum_{\tau \in U_n(k)} (-1)^{\binom{k}{2} + \sum_{i=1}^k |\tau(i)|} \\ &= (-1)^{\binom{k}{2}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} (-1)^{\sum_{j=1}^k i_j}. \end{aligned}$$

From the q -binomial theorem

$$\prod_{i=1}^n (1 + q^i x) = \sum_{k=0}^n q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k,$$

it follows that

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} q^{\sum_{j=1}^k i_j} = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

We deduce that

$$\sum_{\tau \in U_n(k)} \text{sign}(\tau) = (-1)^{\binom{k}{2}} (-1)^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{-1} = (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{-1},$$

so

$$\sum_{\tau \in U_n} \text{sign}(\tau) q^{\text{neg}(\tau)} = \sum_{k=0}^n q^k \sum_{\tau \in U_n(k)} \text{sign}(\tau) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{-1} (-q)^k.$$

From the case $m = 2$ of Corollary 4.2 we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_{-1} = \begin{cases} 0, & \text{if } k \text{ and } n - k \text{ are odd;} \\ \begin{pmatrix} \lfloor n/2 \rfloor \\ \lfloor k/2 \rfloor \end{pmatrix}, & \text{otherwise.} \end{cases}$$

Thus, for n even ($n = 2m$):

$$\sum_{\tau \in U_n} \text{sign}(\tau) q^{\text{neg}(\tau)} = \sum_{t=0}^m \binom{m}{t} (-q)^{2t} = (1 + q^2)^m,$$

and for n odd ($n = 2m + 1$):

$$\sum_{\tau \in U_n} \text{sign}(\tau) q^{\text{neg}(\tau)} = (1 - q) \sum_{t=0}^m \binom{m}{t} (-q)^{2t} = (1 - q)(1 + q^2)^m.$$

We conclude that, for n odd ($n = 2m + 1$):

$$\begin{aligned} \sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{flag-major}(\sigma)} &= (1 - q)(1 + q^2)^m [1]_{q^2} [2]_{-q^2} \cdots [2m + 1]_{q^2} \\ &= (1 - q)(1 + q^2)^m \frac{\prod_{t=1}^{2m+1} (1 - q^{2t})}{(1 - q^2)^{m+1} (1 + q^2)^m} \\ &= \frac{(1 - q) \prod_{t=1}^{2m+1} (1 - q^{2t})}{(1 - q^2)^{m+1}} = \frac{\prod_{t=1}^{2m+1} (1 - q^{2t})}{(1 + q)^{m+1} (1 - q)^m} \\ &= [2]_{-q} [4]_q \cdots [2(2m + 1)]_{-q}. \end{aligned}$$

The case of n even is similar. □

6 Other One-Dimensional Characters of B_n

The group B_n has four one-dimensional characters: the trivial character; the sign character; $(-1)^{\text{neg}(\sigma)}$; and the sign of $(|\sigma(1)|, \dots, |\sigma(n)|) \in S_n$, denoted $\text{sign}(|\sigma|)$. We now generalize the results of the previous section to the last two one-dimensional characters.

Theorem 6.1

$$\sum_{\sigma \in B_n} (-1)^{\text{neg}(\sigma)} q^{\text{flag-major}(\sigma)} = [2]_{-q} [4]_{-q} \cdots [2n]_{-q}.$$

Proof. Replace q by $-q$ in Theorem 1.5, and use the fact that the parity of flag-major is equal to the parity of neg. □

Theorem 6.2

$$\sum_{\sigma \in B_n} \text{sign}(|\sigma|) q^{\text{flag-major}(\sigma)} = [2]_q [4]_{-q} \cdots [2n]_{(-1)^{n-1}q}.$$

Proof. Similarly, replace q by $-q$ in Theorem 5.1 and apply the identity $\text{sign}(\sigma) = \text{sign}(|\sigma|) \cdot (-1)^{\text{neg}(\sigma)}$. □

7 Major Index on Subgroups

Let A_n be the group of even permutations on n letters. Then

Corollary 7.1

$$\sum_{\pi \in A_n} q^{\text{maj}(\pi)} = \frac{1}{2} ([1]_q [2]_q \cdots [n]_q + [1]_q [2]_{-q} \cdots [n]_{(-1)^{n-1}q}).$$

Proof. Clearly,

$$\begin{aligned} \sum_{\pi \in A_n} q^{\text{maj}(\pi)} &= \sum_{\pi \in S_n} \frac{1 + \text{sign}(\pi)}{2} q^{\text{maj}(\pi)} \\ &= \frac{1}{2} \left(\sum_{\pi \in S_n} q^{\text{maj}(\pi)} + \sum_{\pi \in S_n} \text{sign}(\pi) q^{\text{maj}(\pi)} \right). \end{aligned}$$

Corollary 3.3 completes the proof.

□

Let B_n^+ be the subgroup of even elements in B_n , D_n the subgroup of elements with even neg (this is a classical Weyl group), and $C_2 \wr A_n$ the subgroup of elements $\sigma \in B_n$ with even $\text{sign}(|\sigma|)$. Then

Corollary 7.2

$$(1) \quad \sum_{\sigma \in B_n^+} q^{\text{flag-major}(\sigma)} = \frac{1}{2}([2]_q[4]_q \cdots [2n]_q + [2]_{-q}[4]_q \cdots [2n]_{(-1)^n q}).$$

$$(2) \quad \sum_{\sigma \in D_n} q^{\text{flag-major}(\sigma)} = \frac{1}{2}([2]_q[4]_q \cdots [2n]_q + [2]_{-q}[4]_{-q} \cdots [2n]_{-q}).$$

$$(3) \quad \sum_{\sigma \in C_2 \wr A_n} q^{\text{flag-major}(\sigma)} = \frac{1}{2}([2]_q[4]_q \cdots [2n]_q + [2]_q[4]_{-q} \cdots [2n]_{(-1)^{n-1} q}).$$

Proof. Theorem 5.1 implies (1), Theorem 6.1 implies (2), and Theorem 6.2 implies (3). □

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